# Calculus

**Following chapters/Subtopics are easy to prepare for getting better success in examination**

**1.Partial Differentiation :**

-Euler's theorem & its corollaries.

-Chain Rule

-Jacobian

-Maxima & Minima of two variables.

-Tangent Plane and Normal line.

**2.Indeterminate forms (All forms ).**

**3.Sequence and Series :**

-Ratio Test.

-Root Test.

- Comparison Test.

-Alternating Series & Leibnitz's Test.

**4.Multiple Integral :**

-Evaluation of Double & Triple integrals.

-Change the order of integration.

-Change of Variables in Double Integrals by Jacobian.

-Area & Volume.

**5.Reduction formulae (All formulae ).**

**6.Tracing of curves:**

-Cartesian Curves : Cissoid of Diocles,Strophoid,Folium of Descartes,

Witch of Agnesi.

-Polar Curves : Cardioid (both cases),Lemniscate of Bernouli,Limacon

# @@@Brief Discussion @@@

## (1) Limit , Continuity and Partial differentiation

***Euler's theorem :**

If u is a homogeneous function of x & y of degree n , then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = nu ……………………………(i)

*to find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ for given function u(x,y) , first find n satisfying u(tx,ty) = $t^n$ u(x,y) and then put value of n in R.H.S. of result (i) .

Cor I : If u is a homogeneous function of x & y of degree n , then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u \quad \ldots\ldots(ii)$$

*to find value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ for given function u(x,y) , first find n satisfying u(tx,ty) = $t^n$ u(x,y) and then put value of n in R.H.S. of result (ii) .

***Modified Euler's theorem(i) :**

If z is a homogeneous function of x & y of degree n & z=f(u) , then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$ ……………………(iii)

*to find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ ,when z is not a homogeneous function of u but it is homogeneous function of x & y( i.e. z=g(x,y)=f(u) ) then find n satisfying g(tx,ty) = $t^n$ g(x,y) also find $f'(u)$and then put value of n , f(u) & $f'(u)$ in R.H.S. of result (iii) .

(ii) If z is a homogeneous function of x & y of degree n & z=f(u) , then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \quad \text{where } g(u) = n\frac{f(u)}{f'(u)} .$$

***Total derivative and chain rule :**

1. Total differential du = $\frac{\partial u}{\partial x}$ dx+ $\frac{\partial u}{\partial y}$ dy .

2. If u=f(x,y) and x=x(t) & y=y(t) then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ .

3. If z=f(x,y) and if x=x(u,v) , y=y(u,v) then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} .$$

4. Derivative of an implicit function , if f(x,y)=0 be an implicit function with y=g(x) ,then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}.$$

***Jacobian:**

1. If u=u(x,y) and v=v(x,y) then the Jacobian of u and v w.r.t. x and y is given by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

To find Jacobian find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and substitutes these values and expand the resulting determinant.

2. If u=u(x,y,z) and v=v(x,y,z) and w=w(x,y,z) then the Jacobian of u,v ,w w.r.t. x,y,z is given by

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$ solve it by similar process as above.

***Properties of Jacobians :**

(i) If u and v are functions of x and y and x and y are functions of r and s, then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,s)}$$

(ii) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$ .

***Maxima and Minima of functions of two variables :**

Step 1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ then equate them to zero and solve these equations.

Let $(a_1, b_1)$ , $(a_2, b_2)$ ………………be the solutions of these equations.

Step 2. Find r=$\frac{\partial^2 f}{\partial x^2}$, s=$\frac{\partial^2 u}{\partial x \partial y}$, t=$\frac{\partial^2 f}{\partial y^2}$ at these points .

Step 3. If rt-$s^2$> 0 and

(a) If r>0 or t>0 at one or more points then those points are the points of minima.
(b) If r<0 or t<0 at one or more points then those points are the points of maxima.

Step 4. If rt-$s^2$< 0 then there are no maxima or minima at these points. Such points are called saddle points.

Step 5. If rt-$s^2$= 0 or r=o nothing can be said about the maxima or minima.It requires further investigation.

**<u>Note: The following topics are also important:</u>**

**<u>*Method of obtaining Limit</u>**

Step : 1 Evaluate lim f(x,y) along path I

Step : 2 Evaluate lim f(x,y) along path II

If the limit values along path I and path II are same , then the limit exist otherwise not.

Step : 3 If $x_0$=0 , $y_0$=0 evaluate limit along a path say y=mx or y=m$x^n$

**<u>*Test for Continuity at a point $(x_0, y_0)$ :</u>**

Step : 1 f$(x_0, y_0)$ Should be well defined.

Step : 2 lim f(x,y) as (x,y) →$(x_0, y_0)$ Should exist. ( must be unique and same along any path ).

Step : 3 If $\lim\limits_{(x,y)\to(x_0,y_0)}$ f(x, y),then function is continous at $(x_0, y_0)$ otherwise not.

**<u>* Partial derivative of first order :</u>**

If u = f(x,y,z) then to find partial derivative of u w.r.t. x , differentiate u w.r.t. x only as an ordinary differentiation by keeping y & z constants and is denoted by $\frac{\partial u}{\partial x}$ or $f_x$ .

*If u = f(t) and t = g(x,y) then we can find $\frac{\partial u}{\partial x}=\frac{du}{dt}\frac{\partial t}{\partial x}$ and $\frac{\partial u}{\partial y}=\frac{du}{dt}\frac{\partial t}{\partial y}$ .

**<u>*Which variable is to be regarded as constant (A new notation) :</u>**

Let x = f(r,θ) , y = g(r,θ) , to find $(\frac{\partial x}{\partial r})_y$ , first find relation of x in terms of r & y and then differentiate x partially w.r.t. r keeping y constant.

**<u>*Lagrange's method of undetermined multipliers :</u>**

To find extreme values of a function we consider a function of three variables with one restriction.

i.e.u=f(x,y,z) where ϕ(x,y,z)=0.

Step 1. Form the equation u=f(x,y,z) + λ ϕ(x,y,z) .

Step 2. Write down the equations $\frac{\partial u}{\partial x}$=0 , $\frac{\partial u}{\partial y}$=0 , $\frac{\partial u}{\partial z}$=0

Step 3. Solve these resulting equations along with ϕ(x,y,z)=0 and find the values of x,y,z and λ .

The values of x,y,z so obtained will give the extreme value of f(x,y,z).

***Taylor's expansions for functions of two variables :**

f(x+h,y+k) = f(x,y) + $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)$f + $\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2$f + $\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3$f + .....................

Cor.1 f(a+h,b+k) = f(a,b) + $\left[hf_x(a,b)+kf_y(a,b)\right]$ + $\frac{1}{2!}\left[h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\right]$ +.....

Cor.2 f(x,y) = f(a,b) + $\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$ +

$\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right]$ + ..................

## (2)Indeterminant forms

L' Hospital's Rule is useful to solve the indeterminant form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

**Note1**. If indeterminate form of the type $0 * \infty$ ,that is

if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then to find its limit, we can proceed as follows.

$$\lim_{x\to a} f(x)\,g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$$

Which is of the type $\frac{0}{0}$, now we can apply L' Hospital's Rule to get the limit.

**Note 2**.

$$\text{If } \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty,$$

$$\text{then } \lim_{x\to a} f(x) - g(x) \text{ is of the form } \infty - \infty$$

and to find its limit we can proceed as follows

$$\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

Which is of the type $\frac{0}{0}$ ,now we can apply L' Hospital's Rule to get the limit.

**Note 3**.

- If we have indeterminate forms of the types $0^0, \infty^0$ or $1^\infty$,
  then in these cases the limit is of the form $\lim_{x \to a} f(x)^{g(x)}$
- To get the solution of it first take $Y = f(x)^{g(x)}$ and then take logarithm on both the sides .
- So, we have $log\, Y = log(f(x)^{g(x)}) = g(x)\, log\,(f(x))$, then apply limit both the sides.
- This process convert, the indeterminant form into the indeterminate form of the type $\frac{0}{0}$.
- Then we can apply L' Hospital's Rule.
- At last take antilogarithm on both the sides to get the limit ( answer will be in power of 'e')

## (3)Convergence of Sequences and Series

**Definition** : let $\{a_n\}_{n=m}^{\infty}$ be a **sequence** . A number L is the limit of $\{a_n\}_{n=m}^{\infty}$ if for every ϵ > 0 there is an integer N such that ,if n ≥ N ,then $|a_n - L| < \epsilon$ .in this case we write

$$\lim_{n \to \infty} a_n = L$$

If such a number L exist ,we say that $\{a_n\}_{n=m}^{\infty}$ converges otherwise we say $\{a_n\}_{n=m}^{\infty}$ diverges .

**Theorem :1 The Sandwich Theorem**

Let $\{a_n\}_{n=m}^{\infty}$ , $\{b_n\}_{n=m}^{\infty}$ and $\{c_n\}_{n=m}^{\infty}$ be sequences of real numbers. If

$$\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n$$

And $a_n \le b_n \le c_n$ then

$$\lim_{n \to \infty} b_n = L$$

**Theorem :2** (i) If $\{a_n\}_{n=m}^{\infty}$ is converges, then $\{a_n\}_{n=m}^{\infty}$ bounded.
(ii) If $\{a_n\}_{n=m}^{\infty}$ is unbounded, then $\{a_n\}_{n=m}^{\infty}$ diverges.

**Theorem : 3** If $\lim_{n \to \infty} |a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$ .

**Note :** $\lim_{n \to \infty} |a_n| = l$ doesn't implies $\lim_{n \to \infty} a_n = l$ .For example $a_n = (-1)^n$.

**Definition** : A series $\sum u_n$ is said to be convergent if $S_n$ ( the sum of first n terms of $\sum u_n$) tends to a finite unique limit " S " as n tends to infinity ,Thus for a convergent series $\lim_{n \to \infty} S_n = S$ .where S is called the sum of the series . if $\lim_{n \to \infty} S_n$ does not exist then $\sum u_n$ is called divergent .

**Theorem :4** if $\sum_{n=1}^{\infty} u_n$ converges then $\lim_{n \to \infty} u_n = 0$ . that is if $\lim_{n \to \infty} u_n \neq 0$ then $\sum_{n=1}^{\infty} u_n$ diverges.

**Theorem :5** The Geometric Series $\sum_{n=0}^{\infty} r^n$ is

1. Convergent if $|r| < 1$ and its sum is $\frac{1}{1-r}$
2. Divergent if $|r| \geq 1$ .
3.

**Theorem :6 The Integral Test**

Let $f(1) + f(2) + f(3) + \cdots . + f(n) + \cdots$ be a nonnegative series and let f be a continuous decreasing function defined on [1,∞). Then the series $\sum_{n=1}^{\infty} f(n)$ converges or diverges according as the integral $\int_1^{\infty} f(x)\ dx$ is finite or infinite .

**Note :** This test is useful when the integration of $f(x)$ is known function . For example the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is given then define $f(x) = \frac{1}{x^p}$, hence using this test we can discuss the convergence of series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

**This test is very useful to solve the example like $\sum_{n=2}^{\infty} \frac{1}{n\ logn}$ , $\sum_{n=1}^{\infty} \frac{1}{n\ l(1+ log^2 n)}$ etc.**

**Theorem :7 The p- series Test**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is convergent if } p > 1 \textit{ and divergent if } P \leq 1$$

**Note : p- series Test will be useful in Comparision Test.**

**Theorem :8 The Comparision Test**

If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be a non-negative series and

(i) If $\sum_{n=1}^{\infty} v_n$ converges and $0 \leq u_n \leq v_n$ for all n ≥ 1, then $\sum_{n=1}^{\infty} u_n$ converges , and $\sum_{n=1}^{\infty} u_n \leq \sum_{n=1}^{\infty} v_n$.

(ii) If $\sum_{n=1}^{\infty} v_n$ diverges and $0 \le v_n \le u_n$ for all n ≥ 1,then $\sum_{n=1}^{\infty} u_n$ diverges .

(iii) If $\lim_{n \to \infty} \frac{u_n}{v_n} = l$ ( finite and non-zero ) ,then both the series converge and diverge together .

**Theorem :9 De Alembert's Ratio Test**

Let $\sum_{n=1}^{\infty} u_n$ be a non-negative series. Assume that $u_n \neq 0$ for all n and $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$

(i) If $0 \le l < 1$ ,then $\sum_{n=1}^{\infty} u_n$ converges

(ii) If $l > 1$ ,then $\sum_{n=1}^{\infty} u_n$ diverges

(iii) If $l = 1$ ,then the test fails.( Apply other known test )

**Key steps:**

- Compare the given series with $\sum_{n=1}^{\infty} u_n$ , find $u_n$ and then replace n by n+1 to get $u_{n+1}$ .
- find $\lim_{n \to \infty} \frac{u_{n+1}}{u_n}$ , call it $l$
- If $0 \le l < 1$ ,then $\sum_{n=1}^{\infty} u_n$ converges
- If $l > 1$ ,then $\sum_{n=1}^{\infty} u_n$ diverges
- If $l = 1$ then the test fails. ( Apply other known test )

**Theorem :10 Cauchy's Root Test**

Let $\sum_{n=1}^{\infty} u_n$ be a non-negative series. Assume that $u_n \neq 0$ for all n and $\lim_{n \to \infty} \sqrt[n]{u_n} = l$

(i) If $0 \le l < 1$ ,then $\sum_{n=1}^{\infty} u_n$ converges

(ii) If $l > 1$ ,then $\sum_{n=1}^{\infty} u_n$ diverges

(iii) If $l = 1$, then use another test to test the convergence.

**Key steps:**

- Compare the given series with $\sum_{n=1}^{\infty} u_n$ find $u_n$ and then replace n by n+1 to get $u_{n+1}$ .
- Find $\lim_{n \to \infty} \sqrt[n]{u_n}$ , call it $l$
- If $0 \le l < 1$ ,then $\sum_{n=1}^{\infty} u_n$ converges
- If $l > 1$ ,then $\sum_{n=1}^{\infty} u_n$ diverges
- If $l = 1$ then use another test to test the convergence

**Theorem :11 Leibnitz's Alternative series Test**

(i) The infinite alternative series $\sum_{n=1}^{\infty} (-1)^n u_n$ converges if each $u_n$ numerically positive and $u_n \ge u_{n+1}$ , for all n ≥ 1 and $\lim_{n \to \infty} u_n = 0$ .

i.e. If (i) Each term is numerically less than its previous term.

(ii) $\lim_{n \to \infty} u_n = 0$

Then the series is convergent. If either condition fails, then it is oscillatory.

**Key steps:**

- Compare the given series with $\sum_{n=1}^{\infty} u_n$ , find $u_n$ and then replace n by n+1 to get $u_{n+1}$
- Each term is numerically less than its previous term.
- Find $\lim_{n \to \infty} u_n$ then The infinite alternative series $\sum_{n=1}^{\infty}(-1)^n u_n$ converges if $\lim_{n \to \infty} u_n = 0$ , otherwise oscillatory.

## (4)Multiple Integral

❖ Evaluation of double integrals when rectangle is given by $a \le x \le b,\ c \le y \le d$:

$$\iint_R f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right]dx = \int_c^d \left[\int_a^b f(x,y)dx\right]dy$$

Step:1 first we have to integrate $f(x,y)$ with respect to y Keeping x as a constant

Step:2 Integrate the resulting function with respect to x.

OR

Step:1 first integrate $f(x,y)$ with respect to x Keeping y as a constant

Step:2 Integrate the resulting function with respect to y

❖ If bounded region R is given by $\mathrm{x} = \mathrm{f}_1(\mathrm{y})$ , $\mathrm{x} = \mathrm{f}_2(\mathrm{y}), \mathrm{y} = \mathrm{c}, \mathrm{y} = \mathrm{d}, (\mathrm{c} < \mathrm{d})$

$$\int_c^d \int_{x=f_1(y)}^{f_2(y)} f(x,y)dA = \int_{y=c}^d \left[\int_{x=f_1(y)}^{f_2(y)} f(x,y)dx\right]dy$$

Step: 1 Draw a strip line parallel to x-axis (horizontal) within the region R. This strip touches the region R on left side to the curve $x = f_1(y)$ and on right side to the curve $x = f_2(y)$. Thus we get inner limits as $x = f_1(y)$ and $\mathrm{x} = f_2(y)$ ( $f_1(y) \le \mathrm{x} \le f_2(y)$).

Step: 2 Take this strip in the region from bottom to top (i.e. from $\mathrm{y} = \mathrm{c}$ to $\mathrm{y} = \mathrm{d}$)

Step: 3 first integrate $f(x,y)$ with respect to x Keeping y as a constant

Step: 4 Integrate the resulting function with respect to y

- If bounded region R is given by $\mathrm{y} = \mathrm{f}_1(\mathrm{x})$ , $\mathrm{y} = \mathrm{f}_2(\mathrm{x}), \mathrm{x} = a, x = b, (a < b)$

$$\int_a^b \int_{y=f_1(x)}^{f_2(x)} f(x,y)dA = \int_a^b \left[ \int_{y=f_1(x)}^{f_2(y)} f(x,y)dy \right] dx$$

Step: 1 Draw a strip line parallel to y-axis (vertical) within the region R. This strip touches the region R at below to the curve $\mathrm{y} = f_1(x)$ and above to the curve $\mathrm{y} = f_2(x)$. Thus we get inner limit as $y = f_1(x)$ and $\mathrm{y} = f_2(x)$ $(f_1(x) \le \mathrm{y} \le f_2(x))$.

Step: 2 Take strip in the region R from left to right (i.e. from $x = a\ to\ x = b$)

Step: 3 first integrate $f(x,y)$ with respect to y Keeping x as a constant

Step: 4 Integrate the resulting function with respect to x

- Evaluation of the double integral in polar form(i.e.in terms of $r\ and\ \theta$ ):

$$\iint_R f(r,\theta)drd\theta = \int_{\theta=\alpha}^{\beta} \left[ \int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)dr \right] d\theta$$

Step: 1 Draw a radial line starting from the origin (pole) and passing through the region R. This line touches the region R to the curves $= f_1(\theta)\ and\ r = f_2(\theta)$ .

Thus the limit is $f_1(\theta) \le r \le f_2(\theta)$

Step:2 $\theta$ varies from the line $\theta = \alpha\ and\ \theta = \beta, so\ \alpha \le \theta \le \beta.$

Step: 3 first integrate $f(r,\theta)$ with respect to $r$ Keeping $\theta$ as a constant

Step: 4 Integrate the resulting function with respect to $\theta$

- Change the order of integration:

Step: 1 Draw a rough sketch of the region R can be found out from the given limits of integration.

Step: 2 If the given limits of inner integration are of $x$(i.e. horizontal strip is given) then using vertical strip, convert the inner limits for $y$. From the points of intersection find limits for x.

$$\int_{c}^{d}\left[\int_{x=f_1(y)}^{f_2(y)} f(x,y)dx\right]dy = \int_{a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)dy\right]dx$$

NOTE: The similar process can be repeated, if the given limits of the inner integration are of y.

## (5)Reduction Formula

$$\int sin^n x\, dx = -\frac{sin^{n-1}x\, cosx}{n} + \left(\frac{n-1}{n}\right)\int sin^{n-2} x\, dx \qquad ,n\epsilon N, n \geq 2$$

$$\int cos^n x\, dx = \frac{cos^{n-1}x\, sinx}{n} + \left(\frac{n-1}{n}\right)\int cos^{n-2} x\, dx \qquad ,n\epsilon N, n \geq 2$$

$$\int_0^{\frac{\pi}{2}} sin^n x\, dx = \int_0^{\frac{\pi}{2}} cos^n x\, dx \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\dots\dots\frac{3}{4}\frac{1}{2}\frac{\pi}{2} & ; n \text{ is an even positive integer} \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\dots\dots\frac{4}{5}\frac{2}{3}1 & ; n \text{ is an odd positive integer} \end{cases}$$

$$\int sin^m x\, cos^n x\, dx = -\frac{sin^{m-1}x cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int sin^{m-2} x\, cos^n x\, dx \quad ; n, m\epsilon N, n \geq 2$$

$$\int_0^{\frac{\pi}{2}} sin^m x\;\; cos^n x\, dx = I_{m,n} = \frac{[(m-1)(m-3)\dots..2 \text{ or } 1][(n-1)(n-3)\dots..2 \text{ or } 1]}{(m+n)(m+n-2)\dots\dots 2 \text{ or } 1}k$$

$$\text{where } k = \begin{cases} \frac{\pi}{2} ; \text{if both } m \text{ and } n \text{ are even positive integars} \\ 1 ; \text{otherwise} \end{cases}$$

$$\int tan^n x\, dx = \frac{tan^{n-1}x}{(n-1)} - \int tan^{n-2} x\, dx \;\; ; \; n\epsilon N, n \geq 2$$

$$\int cot^n x\, dx = -\frac{cot^{n-1}x}{(n-1)} - \int cot^{n-2} x\, dx \;\; ; \; n\epsilon N, n \geq 2$$

$$\int_0^{\frac{\pi}{4}} tan^n x\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cot^n x\, dx = I_n = \frac{1}{n-1} - I_{n-2} \quad \text{and } I_0 = \frac{\pi}{4}$$

$$\int sec^n x\, dx = \frac{sec^{n-2}x\, tanx}{n-1} + \left(\frac{n-2}{n-1}\right)\int sec^{n-2} x\, dx \;\; ; n\epsilon N, n \geq 2$$

$$\int cosec^n x\,dx = -\frac{cosec^{n-2}x\,cotx}{n-1} + \left(\frac{n-2}{n-1}\right)\int cosec^{n-2} x\,dx \;\; ; n\epsilon N, n \geq 2$$

$$\int x^n\,sinmx\;\;dx = -\frac{x^n cosmx}{m} + \frac{n}{m^2}x^{n-1}sinmx - \frac{(n-1)n}{m^2}\int x^{n-2}\sin mx\,dx$$

$$\int x^n cosmx\;\;dx = \frac{x^n sinmx}{m} + \frac{n}{m^2}x^{n-1}cosmx - \frac{(n-1)n}{m^2}\int x^{n-2}\cos mx\,dx$$